United Kingdom
Mathematics Trust

Intermediate Mathematical Olympiad

Maclaurin paper

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Solutions

These are polished solutions and do not illustrate the process of failed ideas and rough work by which candidates may arrive at their own solutions.

It is not intended that these solutions should be thought of as the 'best' possible solutions and the ideas of readers may be equally meritorious.

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1. Solve the pair of simultaneous equations

$$x^2 - 2xy = 1,$$
$$5x^2 - 2xy + 2y^2 = 5.$$

SOLUTION

From the first equation, we have $x^2 = 1 + 2xy$

Substituting this into the second equation, we obtain $5(1 + 2xy) - 2xy + 2y^2 = 5$, which simplifies to $y^2 + 4xy = 0$

Factorising, we have $y(y + 4x) = 0$, and so $y = 0$ or $y = -4x$

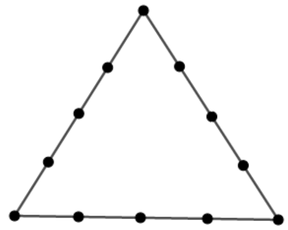
If $y = 0$ we have, from the first equation, $x = \pm 1$

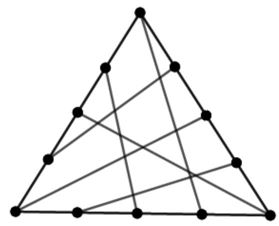
If $y = -4x$ we have, from the first equation, $x = \pm \frac{1}{3}$

So there are four solutions, namely

$(x, y) = (1, 0), (-1, 0), (\frac{1}{3}, -\frac{4}{3}), (-\frac{1}{3}, \frac{4}{3})$

2. The 12 points in the first diagram below are to be joined in pairs by 6 line segments that pass through the interior of the triangle. One example is shown in the second diagram. In how many ways can this be done?



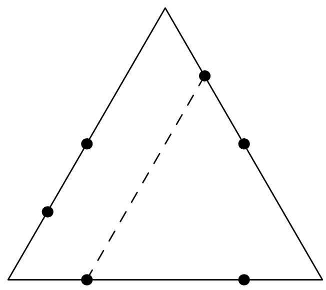


SOLUTION

Call a line segment which obeys the rules a link. There will be six links altogether, since there are twelve points and each link involves two of them. We must ensure that each set of links is counted exactly once. We call a link which involves a vertex a vertex link. Note that there cannot be a link involving two points on the same side.

Each of the three vertices links independently to one of the three points on the side opposite, so there are $3 \times 3 \times 3 = 27$ choices for the vertex links.

We now deal with the six points which remain when the vertex links have been drawn. There are two on each side.



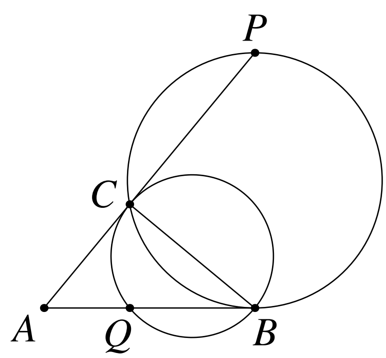
Select the left-hand point on the base. It has 4 possible links, of which one is shown. Note that it cannot link to the other point on the base, since that would not be interior. Now the other point on the base must link to the side with two unused points, since otherwise the final link would not be interior. Hence there are 2 choices here, and altogether there are 4×2 links which involve interior points on the base.

Finally there is no choice about the final link, since only two points remain.

Therefore the total number of ways of choosing the links is $27 \times 8 \times 1 = 216$

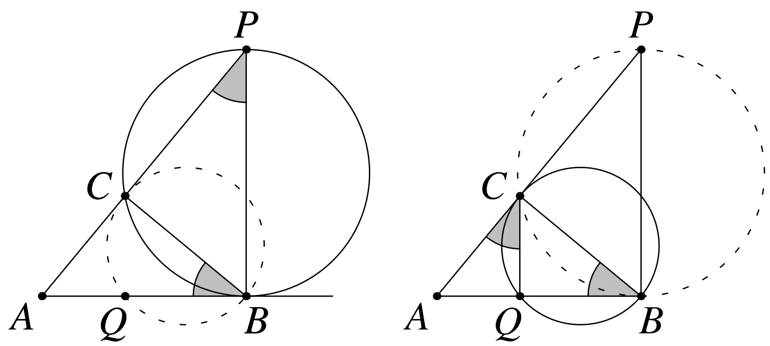
3. The diagram shows a triangle ABC. A circle touching AB at B and passing through C cuts the line AC at P. A second circle touching AC at C and passing through B cuts the line AB at Q.

Prove that $\dfrac{AP}{AQ} = \left(\dfrac{AB}{AC}\right)^3$.



SOLUTION

First solution



By the alternate segment theorem, $\angle BPC = \angle ABC = \angle QCA$

Now the triangles $\triangle ACQ, \triangle ABC$ and $\triangle APB$ are all similar, since they contain the common angle at A and one of the three marked angles.

Hence $\frac{AQ}{AC} = \frac{AC}{AB} = \frac{AB}{AP}$ and so $AP = \frac{AB^2}{AC}$ and $AQ = \frac{AC^2}{AB}$ and so $\frac{AP}{AQ} = \left(\frac{AB}{AC}\right)^3$

Second solution

If we ignore the segment QC we have, by the tangent-secant theorem, $AB^2 = AC \times AP$ and $AC^2 = AQ \times AB$, and now the result is proved by dividing the first by the second.

Remark

The theorem used in the second solution is actually a consequence of the similar triangles in the first solution. It is often phrased in terms of the *power* of the point A with respect to the two circles.

4. A sequence of integers $a_1, a_2, a_3, \ldots$ is defined by

$$a_1 = k,$$
$$a_{n+1} = a_n + 8n \text{ for all integers } n \geq 1.$$

Find all values of k such that every term in the sequence is a square.

SOLUTION

First solution

We have $a_2 = a_1 + 8$ and the first two terms are squares, so we have $8 = b^2 - a^2 = (b+a)(b-a)$ where a and b are positive integers.

Hence either $b - a = 1$ and $b + a = 8$, which is impossible for integers, or $b - a = 2$ and $b + a = 4$, so $a = 1$ and $b = 3$. Hence the only possible value of k is 1.

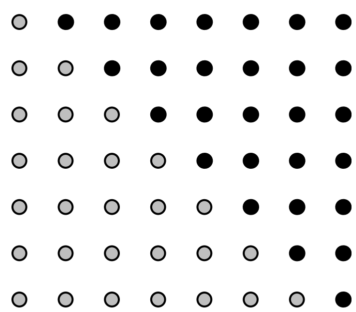
Alternatively, suppose that all the terms are squares, with the first two differing by 8. Note that 16 and 25 differ by 9 and the gaps between successive squares beyond that are shown to be greater than 9. It follows that the first two terms could only be 1, 4, 9 or 16. It is easy to observe that this forces $a_1 = 1$ and $a_2 = 9$, and so the only possible value of k is 1.

Evaluating, we obtain the sequence $1, 9, 25, 49, \ldots$ which are all perfect squares, and we must show that this pattern continues.

From the definition of the sequence, we have $a_n = 1 + 8(1 + 2 + \cdots + (n-1))$, but we need a simpler expression for the sum $1 + 2 + \cdots + (n-1)$. One method is to consider a rectangle of dots divided into two triangles.

The grey dots represent the numbers 1 to 7, and the black dots represent the numbers 7 to 1. Together they form a rectangle with 7×8 dots.

Hence $1 + 2 + 3 + 4 + 5 + 6 + 7 = \frac{1}{2}(7 \times 8)$ and in general $1 + 2 + \cdots + (n-1) = \frac{1}{2}(n-1)n$



As a consequence we have $a_n = 1 + 4(n-1)n = (2n-1)^2$, which is a perfect square.

Second solution

We have $a_n = k + (1 + 2 + \cdots + (n-1))$ and so $a_n = k + 4(n-1)n = 4n^2 - 4n + k$, as above, but for general k. Rearrange this to obtain $a_n = (2n-1)^2 + k - 1$

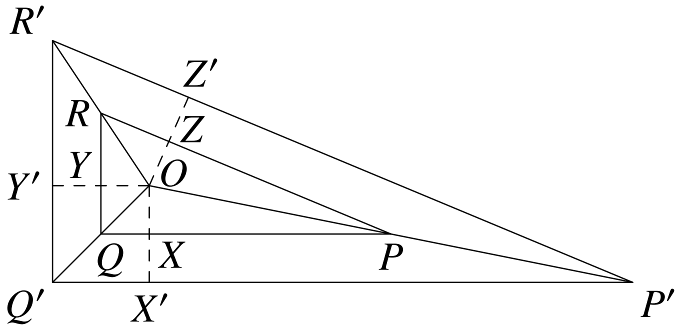
If a_n is always square, then we have an infinite number of pairs of squares which differ by a fixed value $k - 1$. However this cannot happen if $k - 1 > 0$, since the gap between consecutive squares N^2 and $(N+1)^2$ is $2N + 1$, which will eventually be larger than $k - 1$ for a fixed $k \geq 2$

Hence $k = 1$ and $a_n = (2n-1)^2$

5. A triangular playground has sides, in metres, measuring 7, 24 and 25. Inside the playground, a lawn is designed so that the distance from each point on the edge of the lawn to the nearest side is 2 metres. What is the area of the lawn?

SOLUTION

First solution



Since the corresponding sides of the two triangles are parallel, there is a centre of enlargement O so that $\frac{OP}{OP'} = \frac{OQ}{OQ'} = \frac{OR}{OR'} = k$

The same is true for the perpendiculars from O to the sides, so $\frac{OX}{OX'} = \frac{OY}{OY'} = \frac{OZ}{OZ'} = k$

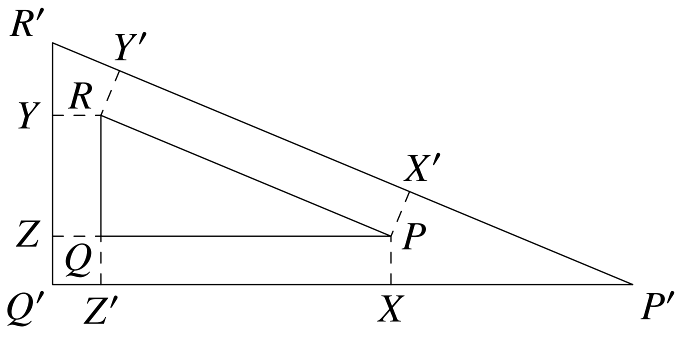
The width of the path is constant, so $XX' = YY' = ZZ'$ and hence $OX = OY = OZ$

It follows that O is the incentre of both $\triangle PQR$ and $\triangle P'Q'R'$

The area of $\triangle P'Q'R'$ is 84 and its semiperimeter is 28. Using the formula Area$(\triangle) = rs$ we have $r = 3$

The width of the path is 2, so $OX = 1$ and $k = \frac{1}{3}$. Hence the area of the lawn is $\frac{28}{3}$

Second solution



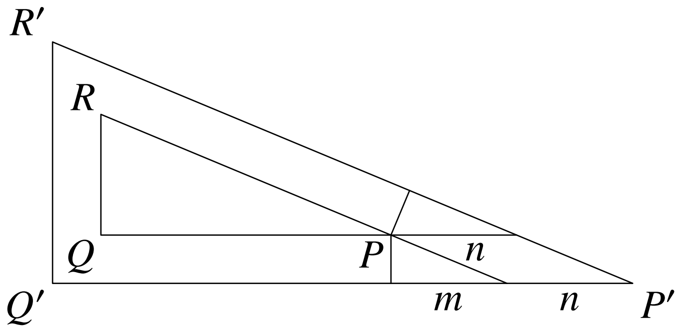
Let the ratio of the lengths of the sides of the small triangle to the lengths of the sides of the large triangle be k. Then we have $PQ = 24k, QR = 25k, RP = 7k$ and also $P'Z = P'Z' = 2$

Hence $XQ' = X'Q' = 22 - 24k$ and $YR' = Y'R' = 5 - 7k$

Now, evaluating $X'R'$ in two ways, we have $25 - (5 - 7k) - (22 - 24k) = 25k$ and so $k = \frac{1}{3}$

The rest of the proof follows as in the first solution.

Third solution



This approach focuses on the two small triangles with a vertex at P, which are similar to $P'Q'R'$ with scale factor $\frac{2}{7}$

Hence $m = \frac{48}{7}$ and $n = \frac{50}{7}$

However, the parallelogram with opposite vertices at P and P' is a rhombus, and now we calculate PQ as $24 - 2 - m - n = 8$. The rest follows.

6. A cat and a mouse occupy the top right and bottom left cells respectively of an $m \times n$ rectangular grid, where $m, n > 1$. Each second they both move diagonally one cell.

For which pairs (m, n) is it possible for the cat and the mouse to occupy the same cell at the same time?

Note: For every pair (m, n) you must either prove that it is impossible for the cat and the mouse to occupy the same cell at the same time, or explain why there is a sequence of moves that ends with the cat and the mouse occupying the same cell at the same time.

SOLUTION

Colour the grid with alternating black and white squares (like a chessboard) and note that diagonally adjacent cells have the same colour.

When $m + n$ is odd, the top left and bottom right squares have different colours, so the cat and mouse never meet.

We now examine what happens when $m + n$ is even, and so m and n are either both even or both odd.

First suppose that m and n are both even. We indicate the parity of the length of any path. The example below shows what happens with $m = 6$ and $n = 4$, with the cat on the left and the mouse on the right. Note that the parity changes with each move. This is not affected by the particular even numbers chosen.

	1		1		1
0		0		0	
	1		1		1
0		0		0	

	0		0		0
1		1		1	
	0		0		0
1		1		1	

In order for the cat and mouse to meet, there must be a pair of corresponding squares in both grids with the same parity. This is not the case, so they cannot meet.

If, now, both m and n are odd – for example with $m = 5$ and $n = 3$ – the argument above does not work.

0		0		0
	1		1	
0		0		0

0		0		0
	1		1	
0		0		0

However, we can show that they can meet. One easy way to do this is for the mouse to go back and forth from the bottom corner square, while the cat zigzags towards it. Assuming that $m \geq n$, they will both arrive there after $m - 1$ moves and they can chat about old times until the cat feels hungry. A similar argument holds if $m < n$. Of course, if $m = n$ they can go directly to the centre square and meet there, a more democratic outcome.